ON THE LENGTH AND NUMBER OF SERVED CUSTOMERS OF THE BUSY PERIOD OF A GENERALISED $M/G/1$ QUEUE WITH FINITE WAITING ROOM

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Reprinted from the
ADVANCES IN APPLIED PROBABILITY
Vol. 5, No. 2, pp. 379–389
August 1973

Adv. Appl. Prob. **5**, 379–389 (1973)
Printed in Israel

ON THE LENGTH AND NUMBER OF SERVED CUSTOMERS OF THE BUSY PERIOD OF A GENERALISED $M/G/1$ QUEUE WITH FINITE WAITING ROOM

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Abstract

Customers arrive in groups to a single server queue with finite waiting room. Two-dimensional distributions for times and numbers of served customers between occurrences of states in the embedded Markov chain are obtained by linear algebra giving systems of equations for joint Laplace-Stieltjes transforms. For $M/M/1$ a simple recursion relation for the joint transform of the two variables in the title is derived and used to obtained the first and second moments.

QUEUE; FINITE WAITING ROOM; GENERALISED POISSON PROCESS; EMBEDDED MARKOV CHAIN; BUSY PERIOD LENGTH; JOINT TRANSFORMS; LINEAR EQUATIONS FOR TRANSFORMS; LIMIT THEOREM FOR BUSY PERIOD LENGTH

1. Introduction

We will treat a service system with finite waiting room and such that the customer input is a generalised Poisson process. The joint distribution of the time and the number of served customers between occurrences of states in the embedded Markov chain of the system is studied. Special attention will be given to transitions from j customers to k with $j > k$. Cohen [1] and Tomko [5] have studied the busy period for ordinary Poisson input. Cohen uses a branching argument and integration in the complex plane. A formula for the Laplace-Stieltjes transform of the busy period distribution is obtained using contour integrals. Tomko uses real analysis and gives a limit theorem for the busy period distribution, which in the present paper is generalised to a two-dimensional form. The generating function

$$\sum_{N=1}^{\infty} \mu_N z^N = \mu z / \{\hat{F}(\lambda(1-z)) - z\},$$

where μ_N is the mean of a busy period given N waiting places, F the service time d.f., $\hat{F}$ its Laplace-Stieltjes transform, μ its mean and λ the intensity of arrival,

Received in revised form 23 November 1972.

follows from the work of both authors and is used by Tomko to obtain μ_N for some special cases.

In this paper we use real analysis and an argument more related to Tomko's than to Cohen's. The transforms and limit results are obtained by linear algebra using matrices related to the transition probability matrix of the embedded Markov chain.

The basic idea of our method of deriving the recursive relation (2) is due to Råde ([3], p. 452).

2. Linear equations for transforms

Groups of customers arrive in a Poisson process with intensity λ. A group contains j individuals with probability p_j, $j = 1, 2, \cdots$. If the queue capacity is insufficient, then as many customers as possible join the queue while the rest are lost. A finite system size $n \geqq 2$ ($n-1$ waiting places) is supposed. The successive service times are independent and have distribution function F. Consider the system at an instant when the service of a customer has just ended and j customers are in the system, not including the one just leaving, $j = 0, 1, \cdots, n-1$. Let $Y_{j,k}^{(n)}$ be the time that elapses until for the first time again k customers are in the system right after the service of a customer, $k = 0, 1, \cdots, n-1$, and let $X_{j,k}^{(n)}$ be the number of service phases during this time.

Now set

$$p_u(z) = \exp\left\{-\lambda u \left(1 - \sum_{j=1}^{\infty} p_j z^j\right)\right\},$$

which represents the pr.g.f. of the number of arriving customers in an interval of length u;

$$v_r^{(n)}(u) = \begin{cases} p_u^{(r)}(0)/r!, & r = 0, 1, \cdots, n-1, \\ \sum_{k=n}^{\infty} p_u^{(k)}(0)/k!, & r = n, \\ 0, & \text{elsewhere}; \end{cases}$$

and

$$g_{r,n}(s) = \int_0^\infty e^{-su} v_r^{(n)}(u)\, dF(u).$$

The Laplace-Stieltjes transform of F is

$$\hat{F}(s) = g_{0,0}(s) = \int_0^\infty e^{-su} dF(u).$$

We take interest first in

$$F_{j,k}^{(n)}(t,x) = P(\{Y_{j,k}^{(n)} \leqq t\} \cap \{X_{j,k}^{(n)} = x\}).$$

By conditioning with respect to the time u of the service of the first customer, to the number r of admitted customers during this time, and by iterating from u, using the Markov property, we get

$$\text{(1)} \quad \begin{aligned} F_{j,k}^{(n)}(t,1) &= \int_0^t v_{k+1-j}^{(n-j)}(u)\, dF(u), \quad j \geqq 1, \\ F_{j,k}^{(n)}(t,x) &= \int_0^t \sum_{\substack{r=0 \\ r \neq k+1-j}}^{n-j} v_r^{(n-j)}(u)\, F_{r+j-1,k}^{(n)}(t-u, x-1)\, dF(u), \quad j \geqq 1, x \geqq 2. \end{aligned}$$

Define

$$\hat{F}_{j,k}^{(n)}(s,x) = \int_0^\infty e^{-st} d_t F_{j,k}^{(n)}(t,x).$$

Then (1) gives

$$\text{(2)} \quad \begin{aligned} \hat{F}_{j,k}^{(n)}(s,1) &= g_{k+1-j,n-j}(s), \quad j \geqq 1, \\ \hat{F}_{j,k}^{(n)}(s,x) &= \sum_{\substack{r=0 \\ r \neq k+1-j}}^{n-j} g_{r,n-j}(s)\, \hat{F}_{r+j-1,k}^{(n)}(s, x-1), \quad j \geqq 1, \ x \geqq 2. \end{aligned}$$

Now we may obtain the transform

$$\hat{\hat{F}}_{j,k}^{(n)}(s,z) = E(\exp\{-sY_{j,k}^{(n)}\} z^{X_{j,k}^{(n)}}) = \sum_{x=1}^{\infty} z^x \hat{F}_{j,k}^{(n)}(s,x).$$

We will in the sequel consider only $s \geqq 0$ and $0 \leqq z \leqq 1$. Putting $z = e^{-w}$ we get the two-dimensional Laplace-Stieltjes transform of $(Y_{j,k}^{(n)}, X_{j,k}^{(n)})$. Now (2) gives

$$\hat{\hat{F}}_{j,k}^{(n)}(s,z) = z g_{k+1-j,n-j}(s) + z \sum_{\substack{r=0 \\ r \neq k+1-j}}^{n-j} \sum_{x=2}^{\infty} z^{x-1} g_{r,n-j}(s) \hat{F}_{r+j-1,k}^{(n)}(s, x-1), j \geqq 1.$$

For $j = 0$ we have to use a special argument, looking at the time to the next arrival and the number of customers then arriving. We obtain

$$\text{(3)} \quad \begin{aligned} \hat{\hat{F}}_{0,k}^{(n)}(s,z) &= \sum_{j=1}^{n-1} \frac{\lambda p_j}{s+\lambda} \hat{\hat{F}}_{j,k}^{(n)}(s,z) \\ &\quad + \left(\sum_{j=n}^{\infty} p_j\right) \frac{\lambda z \hat{F}(s)}{s+\lambda} [\min(1, n-k-1)(\hat{\hat{F}}_{n-1,k}(s,z) - 1) + 1], \end{aligned}$$

$$\hat{\hat{F}}_{j,k}^{(n)}(s,z) = z g_{k+1-j,n-j}(s) + \sum_{\substack{r=0 \\ r \neq k+1-j}}^{n-j} z g_{r,n-j}(s)\, \hat{\hat{F}}_{r+j-1,k}^{(n)}(s,z), \quad 1 \leqq j \leqq n-1.$$

The case $k = n - 1$ is special, as is seen. For every fixed k (3) defines a linear system of equations of order n. With matrix formulation we write

$$\hat{\hat{F}}_k^{(n)}(s,z) = A_k^{(n)}(s,z)\,\hat{\hat{F}}_k^{(n)}(s,z) + c_k^{(n)}(s,z).$$

Here

$$\hat{\hat{F}}_k^{(n)} = [\hat{\hat{F}}_{0,k}^{(n)}, \cdots, \hat{\hat{F}}_{n-1,k}^{(n)}]^T$$

and

$$c_k^{(n)} = [b(n,k), zg_{k,n-1}, zg_{k-1,n-2}, \cdots, zg_{0,n-k-1}, 0, \cdots, 0]^T,$$

where

$$b(n,k) = [1 - \min(1, n-k-1)]\left(\sum_{r=n}^{\infty} p_r\right)\lambda z\hat{F}(s)/(s+\lambda).$$

The $n \times n$ matrix $A_k^{(n)}$ has element $\{A_k^{(n)}\}_{j\,h}$ in row j and column h, $j, h = 0, 1, \cdots, n-1$, according to

$$\{A_k^{(n)}\}_{j,h} = \begin{cases} \lambda p_h/(s+\lambda),\ j = 0,\ 1 \leqq h \leqq n-2, \\ \dfrac{\lambda p_{n-1}}{s+\lambda} + \min(1, n-k-1)\left(\sum_{r=n}^{\infty} p_r\right), \dfrac{\lambda z\hat{F}(s)}{s+\lambda}, (j,h) = (0, n-1), \\ zg_{h-j+1,n-j}(s),\ j \neq 0,\ h \neq k,\ h \geqq j-1, \\ 0, \text{ elsewhere.} \end{cases}$$

The solution of (3) then becomes

$$\hat{\hat{F}}_k^{(n)}(s,z) = (I - A_k^{(n)}(s,z))^{-1}\, c_k^{(n)}(s,z). \tag{4}$$

To solve (3) one need not, however, use all n equations simultaneously.

$$\hat{\hat{F}}_{k+1,k}^{(n)}, \hat{\hat{F}}_{k+2,k}^{(n)}, \cdots, \hat{\hat{F}}_{n-1,k}^{(n)}$$

may be determined from the last $n - k - 1$ equations of (3). Then one obtains $\hat{\hat{F}}_{0,k}^{(n)}, \cdots, \hat{\hat{F}}_{k\,k}^{(n)}$ by treating the former as constants.

Let $A_N(s,z)$ be the submatrix formed from $A_k^{(n)}$ by using the last N rows and columns, $N = 1, 2, \cdots, n-k-1$. Put $B_N(s,z)$ for the determinant of $I - A_N(s,z)$. Then Cramer's rule and an elementary development of the determinant gives

$$\hat{\hat{F}}_{k+j,k}^{(n)}(s,z) = \frac{z^j \hat{F}(s+\lambda)^j B_{n-k-j-1}(s,z)}{B_{n-k-1}(s,z)},\ j = 1, \cdots, n-k-1. \tag{5}$$

(Put $B_0 \equiv 1$.) By differentiating (3), using, e.g.,

$$\left\{\frac{\partial^2}{\partial s \partial z}\hat{\hat{F}}_{j,k}^{(n)}(s,z)\right\}_{(s,z)=(0,1)} = -E\,(Y_{j,k}^{(n)} X_{j,k}^{(n)}),$$

one may obtain all moments of the distribution of $(Y_{j,k}^n, X_{j,k}^n)$, insofar as they exist, from systems with matrix of coefficients $I - A_k^{(n)}(0,1)$.

3. A limit theorem

We will give an example of the use of (5). We take $k = 0$, which is no restriction as far as the "lower" part of $\hat{\hat{F}}_k^{(n)}$ is concerned. We shall prove the following theorem, for $n \geqq 2$.

Theorem. If $F(0) = 0$, *then*

$$\lim_{\lambda \to \infty} P(\hat{F}(\lambda)^{n-1} X_{j,0}^{(n)} \leqq x) = 1 - e^{-x},\ x \geqq 0.$$

If in addition $\mu = \int_0^\infty t\, dF(t) < \infty$, *then*

$$\lim_{\lambda \to \infty} P(\hat{F}(\lambda)^{n-1} X_{j,0}^{(n)} \leqq x,\ \hat{F}(\lambda)^{n-1} Y_{j,0}^{(n)} \leqq y) = 1 - \exp\{-\min(x, y/\mu)\},\ (x,y) \geqq (0,0).$$

The second limit defines a singular distribution in R^2, an exponential distribution on the half-line $y = \mu x$, $x \geqq 0$.

Proof. Put $a_r = g_{r,r+1}$ and $b_r = g_{r,r}$. We have

$$B_{n-1}(s,z) = \begin{vmatrix} 1 - za_1 & -za_2 & \cdots & -za_{n-2} & -zb_{n-1} \\ -za_0 & 1 - za_1 & \cdots & -za_{n-3} & -zb_{n-2} \\ 0 & -za_0 & \cdots & -za_{n-4} & -zb_{n-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 - za_1 & -zb_2 \\ 0 & 0 & \cdots & -za_0 & 1 - zb_1 \end{vmatrix}.$$

Now we operate on B_{n-1}. First we add to each column all the subsequent columns, using $\sum_{k=r}^{m-1} a_k + b_m = b_r$. Then we subtract from each row the row immediately below, using $b_i - b_{i+1} = a_i$, and obtain

$$B_{n-1}(s,z) = \begin{vmatrix} za_0 & za_1 - 1 & za_2 & \cdots & za_{n-3} & za_{n-2} \\ 0 & za_0 & za_1 - 1 & \cdots & za_{n-4} & za_{n-3} \\ 0 & 0 & za_0 & \cdots & za_{n-5} & za_{n-4} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & za_0 & za_1 - 1 \\ 1 - zb_0 & 1 - zb_0 & 1 - zb_0 & \cdots & 1 - zb_0 & 1 - zb_1 \end{vmatrix}.$$

By developing B_{n-1} along the first column we get

$$B_{n-1}(s,z) = za_0(s)B_{n-2}(s,z) + (-1)^n(1 - zb_0(s))\,D_{n-2}(s,z), \tag{6}$$

where D_{n-2} is the determinant obtained by deleting the first column and the last row. We have

$$za_0(s) = z\hat{F}(s+\lambda)$$

and

$$1 - zb_0(s) = 1 - z\hat{F}(s).$$

Introducing (6) in (5) for $j = 1$ and dividing by $z\hat{F}(s+\lambda)B_{n-2}(s,z)$ we obtain

$$\hat{\hat{F}}_{1,0}^{(n)}(s,z) = \left(1 + \frac{(-1)^n(1 - z\hat{F}(s))\,D_{n-2}(s,z)}{z\hat{F}(s+\lambda)B_{n-2}(s,z)}\right)^{-1} = (1 + d_n(s,z))^{-1}.$$

Put $m = m_n(\lambda) = \hat{F}(\lambda)^{n-1}$. We have

$$\begin{aligned} &d_n(sm, e^{-wm}) \\ &= ((-1)^n D_{n-2}(sm, e^{-wm}))\left(\frac{\hat{F}(\lambda)^{n-2}}{B_{n-2}(sm, e^{-wm})}\right)\left(\frac{\hat{F}(\lambda)}{\hat{F}(sm+\lambda)}\right)\left(\frac{1 - e^{-wm}\hat{F}(sm)}{me^{-wm}}\right). \end{aligned} \tag{7}$$

We have to show that the limit of (7), as $\lambda \to \infty$, is $w + \mu s$. We have $m \to F(0)^{n-1} = 0$, according to assumption. An inspection of the input probabilities $v_r^{(r+1)}(u)$ shows that they all tend to 0 as $\lambda \to \infty$, for fixed $u > 0$. Thus the first factor in (7) tends to 1.

Now we make the induction assumption that the theorem holds for system size from 2 up to $n - 1$. We need the following lemma.

Lemma. If $x = o(\hat{F}(\lambda)^r)$, $\lambda \to \infty$, *then*

$$\lim_{\lambda \to \infty} \hat{F}(\lambda)^r / B_r(sx, e^{-wx}) = 1 \textit{ for } 0 \leqq r \leqq n-2.$$

Proof. We get from (5)

$$\frac{\hat{F}(\lambda)^r}{B_r(sx, e^{-wx})} = \hat{\hat{F}}_{r,0}^{(r+1)}(sx, e^{-wx})\left(\frac{\hat{F}(\lambda)}{\hat{F}(sx+\lambda)}\right)^r e^{rwx}. \tag{8}$$

We have $x < \varepsilon_1 \hat{F}(\lambda)^r$ if $\lambda > K_1$, which together with the induction assumption gives

$$\begin{aligned} \hat{\hat{F}}_{r,0}^{(r+1)}(sx, e^{-wx}) &\geqq \hat{\hat{F}}_{r,0}^{(r+1)}(s\varepsilon_1\hat{F}(\lambda)^r, \exp\{-w\varepsilon_1\hat{F}(\lambda)^r\}) \\ &> [1 + w\varepsilon_1 + \mu s\varepsilon_1]^{-1} - \varepsilon_2 \end{aligned}$$

if $\lambda > K_2$ also. We here allow $\mu = \infty$ only if $s = 0$, corresponding to the first part of the theorem (define $0 \cdot \infty = 0$). Thus the limit of the first factor of (8) is 1.

The mean value theorem gives

$$\hat{F}(sx+\lambda) = \hat{F}(\lambda) + sx\hat{F}'(\lambda + \theta sx),$$

where $0 < \theta < 1$. This shows that the second factor of (8) tends to 1, as does the third of (7).

Now, since $m = o(\hat{F}(\lambda)^{n-2})$, it follows that the second factor of (7) tends to 1. The limit of the fourth factor, finally, is

$$-\frac{\partial}{\partial t}(e^{-wt}\hat{F}(st))_{t=0} = w + \mu s.$$

Thus

$$\lim_{\lambda \to \infty} \hat{\hat{F}}_{1,0}^{(n)}(s\hat{F}(\lambda)^{n-1}, \exp\{-w\hat{F}(\lambda)^{n-1}\}) = [1 + w + \mu s]^{-1}.$$

Now write

$$\begin{aligned} &\hat{\hat{F}}_{j,0}^{(n)}(sm, e^{-wm}) \\ &= \left(\frac{e^{-wm}\hat{F}(sm+\lambda)}{\hat{F}(\lambda)}\right)^{j-1}\left(\frac{\hat{F}(\lambda)^{n-2}}{B_{n-2}(sm, e^{-wm})}\right)\left(\frac{B_{n-j-1}(sm, e^{-wm})}{\hat{F}(\lambda)^{n-j-1}}\right)\hat{\hat{F}}_{1,0}^{(n)}(sm, e^{-wm}), \end{aligned}$$

$j = 1, 2, \cdots, n-1$. Since $m = o(\hat{F}(\lambda)^{n-j-1})$, $1 \leqq j \leqq n-1$, the theorem follows, by the lemma, for $j = 1, 2, \cdots, n-1$. Finally, for $j = 0$ the statement follows easily from the first equation of (3).

If we let $G_n(s,z)$ be the transform of the length of the busy period and the number of served customers in it, we have

$$G_n(s,z) = \lambda^{-1}(s+\lambda)\hat{\hat{F}}_{0,0}^{(n)}(s,z),$$

and obtain the same limiting behaviour for this bivariate distribution.

It remains to show that the induction assumption is valid for $n = 2$. Letting $D_0 = B_0 = 1$, (7) holds also for $n = 2$, and the argument used shows that the theorem holds for $n = 2$ and by induction for all $n \geqq 2$.

Remark. We note that as a byproduct of the development of the limit result, (6) gives a formula for $E(X_{1,0}^{(n)})$ and hence for $E(Y_{1,0}^{(n)}) = \mu E(X_{1,0}^{(n)})$.

We can easily see that $B_N(0,1) = \hat{F}(\lambda)^N$. A little manipulation gives

$$\frac{1 - \hat{\hat{F}}_{1,0}^{(n)}(0,z)}{1 - z} = \frac{(-1)^n D_{n-2}(0,z)}{B_{n-1}(0,z)},$$

and hence, letting $z \to 1-$,

$$E(X_{1,0}^{(n)}) = (-1)^n D_{n-2}(0,1)\hat{F}(\lambda)^{1-n}. \tag{9}$$

We may also be interested in the limiting distribution when $F(0) = p > 0$. Using (5) and (6) we may without great difficulties deduce

$$\lim_{\lambda\to\infty} \hat{F}^{(n)}_{j,0}(s,z) = \frac{(1 - z\hat{F}(s))((zp)^j - (zp)^{n-1}) + (zp)^{n-1} - (zp)^n}{(1 - z\hat{F}(s))(1 - (zp)^{n-1}) + (zp)^{n-1} - (zp)^n},\ j = 1,\cdots,n-1. \tag{10}$$

4. The $M/M/1$ system

We take $p_1 = 1$ and $\hat{F}(s) = \beta/(s+\beta)$, i.e., $M/M/1$ with service intensity β and mean service time $\mu = 1/\beta$. Define as usual $\rho = \lambda/\beta$. Here we have

$$g_{r,r+1} = a_r = \beta\lambda^r(s+\lambda+\beta)^{-(r+1)}.$$

We may for example extract a factor $\lambda/(s+\lambda+\beta)$ from the first row of B_N while developing the determinant at the upper left corner, obtaining

$$B_N(s,z) = B_{N-1}(s,z) - z\beta\lambda(s+\lambda+\beta)^{-2}B_{N-2}(s,z). \tag{11}$$

One can easily solve this difference equation and obtain

$$B_N(s,z) = \frac{(q_1 - z\beta\lambda(s+\beta)^{-1})q_1^N - (q_2 - z\beta\lambda(s+\beta)^{-1})q_2^N}{((s+\lambda+\beta)^2 - 4z\lambda\beta)^{\frac{1}{2}}(s+\lambda+\beta)^N} \tag{12}$$

where $q_1(s,z)$ and $q_2(s,z)$ are the roots of $q^2 - (s+\lambda+\beta)q + z\lambda\beta = 0$, or

$$q_1(s,z) = \tfrac{1}{2}(s+\lambda+\beta + ((s+\lambda+\beta)^2 - 4z\lambda\beta)^{\frac{1}{2}}),$$
$$q_2(s,z) = \tfrac{1}{2}(s+\lambda+\beta - ((s+\lambda+\beta)^2 - 4z\lambda\beta)^{\frac{1}{2}}).$$

Also, if we divide (11) by B_{N-1} we obtain

$$\hat{F}^{(n)}_{k+1,k}(s,z) = z\beta/(s+\lambda+\beta - \lambda\hat{F}^{(n-1)}_{k+1,k}(s,z)),\quad n \geqq k+3, \tag{13}$$

for the time and number of service phases needed to pass from a state in the Markov chain to the state immediately below.

If we now take $k = 0$ and consider

$$G_n(s,z) = \hat{F}^{(n)}_{1,0}(s,z) = E(e^{-sY_n}z^{X_n}),$$

where Y_n is the busy period length and X_n the number of served customers during it, we have

$$\begin{aligned} G_1(s,z) &= z\beta/(s+\beta),\\ G_n(s,z) &= z\beta/(s+\lambda+\beta-\lambda G_{n-1}(s,z)),\qquad n \geqq 2. \end{aligned} \tag{14}$$

The first relation, for no waiting-place, is immediate, and insertion of G_1 in the second shows it to hold true for $n = 2$.

From (14) we may obtain a slightly different explicit formula than the one following from (12). Defining q_1, q_2 as before we get

$$G_n(s,z) = \frac{z\beta[(s+\beta-q_2(s,z))q_1(s,z)^{n-1} - (s+\beta-q_1(s,z))q_2(s,z)^{n-1}]}{(s+\beta-q_2(s,z))q_1(s,z)^n - (s+\beta-q_1(s,z))q_2(s,z)^n}. \tag{15}$$

The formula is correct for $(s,z) \in [0,\infty) \times [0,1]$ except when $\rho = z = s+1 = 1$. In order to calculate moments, it seems most suitable to move the denominator of (14) to the left-hand side and perform the adequate number of differentiations; this will give a relation $a_n = \rho a_{n-1} + b_n$, where a_n is the moment and b_n is known when the lower moments are known. Here we present results for the case $\rho \neq 1$

$$\begin{aligned} E(X_n) &= (1-\rho^n)(1-\rho)^{-1}\\ E(Y_n) &= (1-\rho^n)\beta^{-1}(1-\rho)^{-1}\\ \operatorname{var}(X_n) &= \rho(1+\rho)[1-(2n-1)(1-\rho)\rho^{n-1} - \rho^{2n-1}](1-\rho)^{-3}\\ \operatorname{var}(Y_n) &= ((1+\rho)(1-\rho^{2n}) - 4n(1-\rho)\rho^n)\beta^{-2}(1-\rho)^{-3}\\ \operatorname{cov}(X_n,Y_n) &= (2\rho-(3n-1)\rho^n + (2n-1)\rho^{n+1} + n\rho^{n+2}\\ &\qquad - \rho^{2n} - \rho^{2n+1})\beta^{-1}(1-\rho)^{-3}. \end{aligned} \tag{16}$$

As far as the other variables $(Y^{(n)}_{j,k}, X^{(n)}_{j,k})$ are concerned one may use the representation, not only valid for $M/M/1$, if $j > k$,

$$(Y^{(n)}_{j,k}, X^{(n)}_{j,k}) = \sum_{r=k}^{j-1} (Y^{(n-r)}_{1,0}, X^{(n-r)}_{1,0}),$$

where the terms are independent. This will give, for example,

$$E(X^{(n)}_{j,0}) = j(1-\rho)^{-1} - (\rho^{n-j+1} - \rho^{n+1})(1-\rho)^{-2},\qquad j > 0. \tag{17}$$

For $M/M/1$ the variables considered have an application to the $M/M/m$ system with finite waiting-room, restricted to N places, that is, place for in all $N+m$ customers. If the service time d.f. is $1 - e^{-\alpha t}$, then $Y^{(N+1)}_{1,0}$ with β here equal to αm represents the uninterrupted time when all m servers are busy. This follows from the fact that the smallest of m independent exponentially distributed variables with parameter α is exponentially distributed with parameter αm.

Note that for $\rho < 1$ the limit, as $n \to \infty$, of (14) is equal to its value for an infinite waiting room.

5. A trivariate distribution

The same kind of linear algebra may be used to include also the number of customers not admitted during times between occurrences of states in the Markov chain. We content ourselves with treating the case $k = 0$. For $j = 1,2,\cdots,n-1$, let $Y_j^{(n)}$ be the remaining busy time, $X_j^{(n)}$ the number of served

customers during this time, and $V_j^{(n)}$ the number of customers not admitted during the same time, given that a customer has just left the system and j customers excluding this one are in the system. Having obtained the joint transform for these variables, the transform corresponding to the busy period will follow from an equation similar to the first one of (3). Put

$$\begin{aligned} F_j^{(n)}(t,x,v) &= P(\{Y_j^{(n)} \leqq t\} \cap \{X_j^{(n)} = x\} \cap \{V_j^{(n)} = v\}),\\ \hat{F}_j^{(n)}(s,x,v) &= \int_0^\infty e^{-st} d_t F_j^{(n)}(t,x,v),\\ \hat{\hat{F}}_j^{(n)}(s,z,u) &= E(e^{-sY_j^{(n)}} z^{X_j^{(n)}} u^{V_j^{(n)}}) = \sum_{x=1}^\infty \sum_{v=0}^\infty z^x u^v \hat{F}_j^{(n)}(s,x,v). \end{aligned}$$

We simplify the notation:

$$\begin{aligned} v_r(\tau) &= v_r^{(r+1)}(\tau),\\ g_r(s) &= g_{r,r+1}(s). \end{aligned}$$

n the same way as we derived Equation (1) we obtain

$$\begin{aligned} F_1^{(n)}(t,1,0) &= \int_0^t v_0(\tau)\,dF(\tau),\\ F_j^{(n)}(t,1,v) &= 0,\ (j,v) \neq (1,0),\\ F_j^{(n)}(t,x,v) &= \int_0^t \sum_{\substack{r=0\\ r\neq 1-j}}^{n-j-1} v_r(\tau)F^{(n)}_{r+j-1}(t-\tau,x-1,v)\,dF(\tau)\\ &\quad + \int_0^t \sum_{r=n-j}^{v+n-j} v_r(\tau)F^{(n)}_{n-1}(t-\tau,x-1,v-r+n-j)\,dF(\tau),\ x \geqq 2. \end{aligned} \tag{18}$$

From this relation $F_j^{(n)}$ may be calculated by recursion in x. Taking transforms gives

$$\begin{aligned} \hat{F}_1^{(n)}(s,1,0) &= g_0(s),\\ \hat{F}_j^{(n)}(s,x,v) &= \sum_{\substack{r=0\\ r\neq 1-j}}^{n-j-1} g_r(s)\hat{F}^{(n)}_{r+j-1}(s,x-1,v)\\ &\quad + \sum_{r=n-j}^{v+n-j} g_r(s)\hat{F}^{(n)}_{n-1}(s,x-1,v-r+n-j),\quad x \geqq 2. \end{aligned} \tag{19}$$

By recursion we may obtain $\hat{F}_j^{(n)}$, especially

$$\hat{F}_j^{(n)}(0,x,v) = P(X_j^{(n)} = x,\ V_j^{(n)} = v).$$

From (19) we may now obtain the joint transform in the same way as (3) is

obtained from (2). We get a triple sum, in which we interchange the order of summation between v and r. Leaving out details we state

$$\hat{\hat{F}}_j^{(n)}(s,z,u) = \sum_{r=0}^{n-j-1} zg_r(s)\hat{\hat{F}}^{(n)}_{r+j-1}(s,z,u) + zu^{j-n}\left(\sum_{r=n-j}^{\infty} u^r g_r(s)\right)\hat{\hat{F}}^{(n)}_{n-1}(s,z,u),\quad j = 1,2,\cdots,n-1. \tag{20}$$

Here we define $\hat{\hat{F}}_0^{(n)}(s,z,u) = 1$.

A solution by Cramer's rule will give an expression exactly corresponding to (5). For $M/M/1$ the relation (11) will result again, as will (13). Considering $G_n(s,z,u) = E(e^{-sY_n}z^{X_n}u^{V_n})$, where V_n is the number of customers not admitted during a busy period of the $M/M/1$ system with place for n customers, we obtain the remarkably simple generalisation of (14)

$$\begin{aligned} G_1(s,z,u) &= z\beta(s+\beta+\lambda(1-u))^{-1},\\ G_n(s,z,u) &= z\beta(s+\lambda+\beta-\lambda G_{n-1}(s,z,u))^{-1},\qquad n \geqq 2. \end{aligned} \tag{21}$$

(21) may of course be used to obtain formulas similar to (15) and (16). Differentiation with respect to u gives

$$E(V_n) = \rho^n. \tag{22}$$

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